Management Analytics and Social Insights

www.masi.reapress.com

Manag. Anal. Soc. Insights Vol. 1, No. 1 (2024) 116–128.

Paper Type: Original Article

A Dynamic Systems Perspective on Credit Risk Transfer Bonds Pricing: Insights from Black-Scholes PDE Analysis

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Citation:

Received: 15 January 2024 Revised: 21 April 2024 Accepted: 10 June 2024	Azimi, S., Dabaghian Amiri, M., & Waloo, S. (2024). A Dynamic Systems Perspective on Credit Risk Transfer Bonds Pricing: Insights from Black-Scholes PDE Analysis. Management Analytics and Social Insights,1(1), 116-128.

Abstract

This study delves into the dynamic systems paradigm applied to Black- Scholes partial differential equations (PDEs) and examines its efficacy in pricing credit risk derivatives. The Black-Scholes PDE is a cornerstone in financial research for pricing various instruments. However, the intricacies inherent in many financial scenarios often preclude the derivation of closed-form solutions, necessitating recourse to numerical methods for solutions. In response to this challenge, our paper proposes a novel approach leveraging dynamic systems derived from a semi-discretization technique applied to the price variable within the Black-Scholes PDE framework. Our focus extends to credit risk transfer bonds, a derivative instrument designed to reallocate credit risk from a financial institution to a risk buyer or investor. By delineating the structural characteristics of these bonds, we construct a PDE analogous to the Black- Scholes equation to capture the pricing dynamics inherent in such instruments. Subsequently, we employ the dynamic systems technique to solve the credit risk transfer bond PDE, thereby facilitating a comprehensive examination of the pricing dynamics involved. The application of this methodology is demonstrated through the utilization of PYTHON software, which enables the visualization and analysis of the obtained results. By bridging the gap between dynamic systems theory and financial derivatives pricing, this research advances computational finance methodologies and offers insights into the pricing mechanisms of credit risk transfer bonds within a dynamic systems framework.

Keywords: Dynamic System, Black Scholes PDE, Credit Risk Derivative, Credit Risk Transfer, Credit Risk Transfer Bond.

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1|Introduction

In this paper, we intend to solve the Black Scholes PDE using a new approach to Azizi et al. [1] dynamical system method. We want to provide a solution model for PDE by dynamic systems and utilize it in pricing a specific type of financial instrument. In other words, after introducing the dynamical system approach, this method will be used to price a bond that covers the credit risk in a contract. Hence, the introduction section is divided into two general subsections: the first part for dynamical systems and the second part for credit risk bonds.

1.1|Dynamic system and PDE

Dynamic systems are very practical topics in various sciences and can help solve many engineering, differential equations, statistics, and other problems. Also, there are a lot of applications in financial concepts [2, 3, 4].

In this study, we intend to find a very simple solution for Black Scholes heat PDE by using dynamic systems, and since such studying is very low in Black Scholes PDE, our method is practical and new. There have been many studies on the integration of dynamic systems and PDE, but in this article, we want to describe some of the studies that have been influential in our research. One of the most important was the paper of Vázquez and Vitillaro, which gave us a view. Vázquez and Vitillaro obtained a system of equations from a heat equation, and some other conditions and then proved that the answer is well-posed in the domain [5]. Schmitt et al. have studied Systems of Partial Differential Equations and provided a solution to the problem of writing software that attains maximum performance and scalability while remaining portable and easily composable [6]. Cheng also estimates the flow of ocean currents by combining dynamic systems with PDE [7]. Additionally, [8] and [9] illustrate the utilization of dynamical systems in financial applications.

In this study, Black Scholes PDE, as written below, converts to a dynamic system by discretizing the spatial variable, and the solution is obtained with the dynamic system method. This method is very simple but highly efficient, so it can be very practical in different research.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

where r and σ are constant and interest rate and volatility, respectively. S is the price of the underlying asset and a random variable where its dynamic is as follows:

$$dS = rSdt + \sigma S dw_t. \tag{2}$$

w_t is a standard Wiener process. Also $V(S,t)$ is the value of a financial instrument depending on the price of the underlying asset and time.

1.2|Credit Risk Transfer

Every financial institution seeks to increase its sources of income for greater profitability. For example, the bank tries to obtain profit from someone by lending and borrowing money and increasing these activities to make more profit. To increase profits, a second party, a contract or borrower, may time-consuming its obligation to perform a series of monetary flows for various reasons, causing the financial institution to suffer a loss. Although there may be guarantees, getting the process in place can be time-consuming and costly. In general, the economic system and especially banks face various forms of risk, among which four types of credit risk, operational risk, market risk, and liquidity risk, cause the most damage to the body of banks. Therefore, it is very important to adopt and design a preventive situation, reduce some or all of these risks, and prevent much damage in part or whole. Let's study one of these risks and offer a solution for it.

As we have said, credit risk is one of the risks that a financial institution may suffer. Credit risk, in general, is a risk that affects the achievement of a financial institution or its shareholders by weakening the debtor's creditworthiness, which more precisely refers to the debtor's inability to meet its obligation. Eliminating such risks or avoiding items that involve such risks is inevitable; this will keep the financial institution away from a large financial source, and it is not a reason not to use it, but instead, solutions can be used to solve such problems. Financial markets typically use instruments called credit risk transfer (CRT) to solve these problems. In general, there are several types of CRT tools called credit derivatives, which are as follows:

- Credit default swaps
- Collateralized debt obligations
- Total return swaps
- Credit default swap options
- Credit spread forwards

In this study, we intend to model a bond to cover credit risk. This bond can be part of the Collateralized debt obligations, and we consider it a credit risk transfer bond (CRTB). Not considering credit risk in modeling can lead us to make strategic mistakes in many cases, so considering this element will bring our calculations closer to the real value. Among the cases where the use of credit risk is option pricing, the first research in this area is Johnson and Stulz's study [10], and we consider this approach in our modeling in this research. There are many models for credit risk, but in this study, we will use the Klein credit risk approach [11]. Many studies have used the Klein approach. For example, Kim and Koo used this kind of modeling in exchange option pricing and provided a closed-form using Mellin transform [12]. They also implicitly used this concept in put option pricing with default risk [13]. Also, Ma and Xiao, when the underlying asset price follows the GARCH diffusion model with a random interest rate, addressed the issue of pricing European vulnerable options using the Fourier transform [14].

In this paper, we present a method for solving Black Scholes PDE using dynamic systems and utilize it in CRTB pricing. For a better explanation of this process, we have dedicated the next section () to Dynamic Systems and Black Scholes PDE, and in section , the CRTB mechanism and its pricing model are introduced. Eventually, in section , we use the section solution method for the CRTB pricing model and show the results by PYTHON software.

2|Dynamic system approach in Black Scholes PDE

There are many methods for solving Black Scholes PDE, for example, finite element, finite difference, meshfree method, or even obtaining close-form. When this PDE shows the call and put option prices, its close form can be as follows:

$$\begin{array}{l} C(S_t,t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \\ P(S_t,t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t \\ d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{array}, \tag{3}$$

where T, K, r, and σ are maturity time, strike price, interest rate, and volatility, respectively, and they are constant. Also, S is the price of the underlying asset, t is the time, and $P(S,t)$ and $C(S,t)$, which sit instead of V in equation 1, are call and put option prices, respectively.

Black Scholes PDE is seen in many financial models. Sometimes, the condition of the problem is such that no close form can be found for it, so other methods, especially numerical methods, causes are used. In this section, we want to provide a solution for this PDE, and we need to introduce the solution of homogeneous and inhomogeneous linear dynamic systems.

2.1|Homogeneous and inhomogeneous linear dynamic systems

Dynamic systems, like other mathematical disciplines, try to analyze various phenomena. What sets the dynamic system apart from other disciplines is its approach to analysis. Sometimes the studied phenomena are linear homogeneous, which can be written as below:

$$\dot{X} = A_{n\times n}X, \tag{4}$$

where A is a square matrix, $X = (x_1, \ldots, x_n) \in R^n$ is a vector and $\dot{X} = (\frac{d}{dt}x_1, \ldots, \frac{d}{dt}x_n)$ is a derivative of X respect to the time. By having the initial conditions X_0, the solution of the system is as

$$X = e^{At}X_0. \tag{5}$$

e^{At} can be considered using a simple Taylor series at $t = 0$ as follows:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + ... \tag{6}$$

The problem situation may also lead to an inhomogeneous linear dynamic system as

$$\dot{X} = A_{n\times n}X + B_{n\times 1}(t), \tag{7}$$

that for solving, we have the following famous theorem.

Theorem 1. *If $\phi(t)$ be a solution of the dynamic system $\dot{X} = A_{n\times n}X$, then the system solution of $\dot{X} = A_{n\times n}X + B_{n\times 1}(t)$ with the initial value $X(t_0) = X_0$ is*

$$X(t) = \phi(t)\phi^{-1}(0)X_0 + \int_0^t \phi(t)\phi^{-1}(z)B(z)dz. \tag{8}$$

Proof: Suppose $X(t) = \phi(t)\phi^{-1}(0)X_0 + \int_0^t \phi(t)\phi^{-1}(z)B(z)dz$ is correct, so it must be true in $\dot{X} = A_{n\times n}X + B_{n\times 1}(t)$. Applying the derivative on $X(t)$, the $\dot{X}(t)$ is

$$\dot{X}(t) = \dot{\phi}(t)\phi(0)X_0 + \phi(t)\phi^{-1}(t)B(t) + \int_0^t \phi(t)\phi^{-1}(z)B(z)dz. \tag{9}$$

According to the theorem assumption, we have $\dot{\phi}(t) = A\phi(t)$, so

$$\begin{aligned}\dot{X}(t) &= A[\phi(t)\phi(0)X_0 + \int_0^t \phi(t)\phi^{-1}(z)B(z)dz] + B(t)\\ &= AX(t) + B(t),\end{aligned} \tag{10}$$

which shows the correctness of the theorem and the proof is complete. □

2.2|Black Scholes PDE solution

As mentioned, Black Scholes PDE is a kind of heat equation that is used in many financial pricing issues, sometimes, we can not reach a close-form for it according to the condition of the problem and designing a method for obtaining the solution easier and faster in very useful and suitable so in this section, we explain a new advantageous method. For better modelling, the pricing model 1 is considered with a variable change of $\tau = T - t$ as follows:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \tag{11}$$

Now to apply the dynamic system, we discretize it respect to S. In fact, depending on the type of financial issue, we can consider the range of changes of S as a finite interval, divide it equally into $n+2$ parts as $S_0=0, S_1, S_2, ..., S_n, S_{n+1}=S_{\max}$ and using this discretization and finite differences, the derivatives respect to S is as follows assuming $V(S_i,\tau)=V_i(\tau)$:

$$\begin{array}{c} \frac{\partial V(S_i,\tau)}{\partial S} = \frac{\partial V_i(\tau)}{\partial S} \simeq \frac{V_{i+1}(\tau)-V_{i-1}(\tau)}{2dS} \\ \frac{\partial^2 V(S_i,\tau)}{\partial S^2} = \frac{\partial^2 V_i(\tau)}{\partial S^2} \simeq \frac{V_{i+1}(\tau)-V_i(\tau)+V_{i-1}(\tau)}{dS^2} \end{array} \tag{12}$$

where

$$dS = S_{i+1} - S_i \ , \ i \in \{0,1,...,n\} \tag{13}$$

Now by placing the derivatives 12 in the PDE 11, we have

$$\dot{V}(\tau) = AV(\tau) + B(\tau) \tag{14}$$

V is the price vector, $\dot{V}$ is the derivative of V with respect to τ, A is a tridiagonal derivatives matrix and $B(\tau)$ is boundary conditions as follows:

$$\begin{array}{cc} A = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_3 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix}, & B(\tau) = \begin{bmatrix} a_1 V_0(\tau) \\ 0 \\ \vdots \\ 0 \\ c_n V_{n+1}(\tau) \end{bmatrix}, \\ V = \begin{bmatrix} V_1(\tau) \\ V_2(\tau) \\ \vdots \\ V_{n-1}(\tau) \\ V_n(t) \end{bmatrix}, & \dot{V} = \begin{bmatrix} \frac{d}{dt}V_1(\tau) \\ \frac{d}{dt}V_2(\tau) \\ \vdots \\ \frac{d}{dt}V_{n-1}(\tau) \\ \frac{d}{dt}V_n(\tau) \end{bmatrix} \end{array} \tag{15}$$

where

$$\begin{array}{l} a_i = \frac{0.5\sigma^2 \mathrm{S}_i^2}{dS^2} - \mathrm{r}\frac{S_i}{2dS} \\ b_i = -\frac{\sigma^2 \mathrm{S}_i^2}{dS^2} - r \\ c_i = \frac{0.5\sigma^2 \mathrm{S}_i^2}{dS^2} + \mathrm{r}\frac{S_i}{2dS} \end{array} \tag{16}$$

2.3|Stability

In this section, we examine the stability of the solution. Since $V(S,\tau)$ is a two-varying function, the stability of this function can be examined from both t and S perspectives. Since the function's behavior when S is infinitely infinite is not important in these cases, we will only examine the stability of t. For this purpose, we will first define stability; then we will examine stability using the problem properties.

Definition 1. *(Lyapunov stability)*[15]*: The solution $X(t)$ is a Lyapunov stable if for each ε, there exists a δ that for other solution $Y(t)$, satisfying $|X(t_0)-Y(t_0)|<\delta$, we can draw conclusions $|X(t)-Y(t)|_{t>t_0}<\varepsilon$.*

According to the definition, we assume that V is the solution to a linear system with the initial condition V_0. Also, suppose F_0 was obtained from perturbation of V_0, and the solution is F.

Assuming that the boundary conditions are equal for two problems, we do not consider the boundary conditions without losing the generality of the problem, so:

$$\begin{array}{c} V - F = e^{A\tau}V_0 - e^{A\tau}F_0 \\ e^{A\tau}(V_0 - F_0). \end{array} \tag{17}$$

Using Norm, we can reach to$\|V-F\| < L\,\|(V_0-F_0)\|$ where $\left\|e^{A\tau}\right\| = L$. Now, considering $\|V-F\| < L\,\|(V_0-F_0)\| \le \varepsilon$ and $\|(V_0-F_0)\| < \delta$, we can be analyzed that for every ε, there exists a δ as $\delta = \varepsilon/L$ that the above relation can be established in it. In other words, it indicates the Lyapunov's stability.

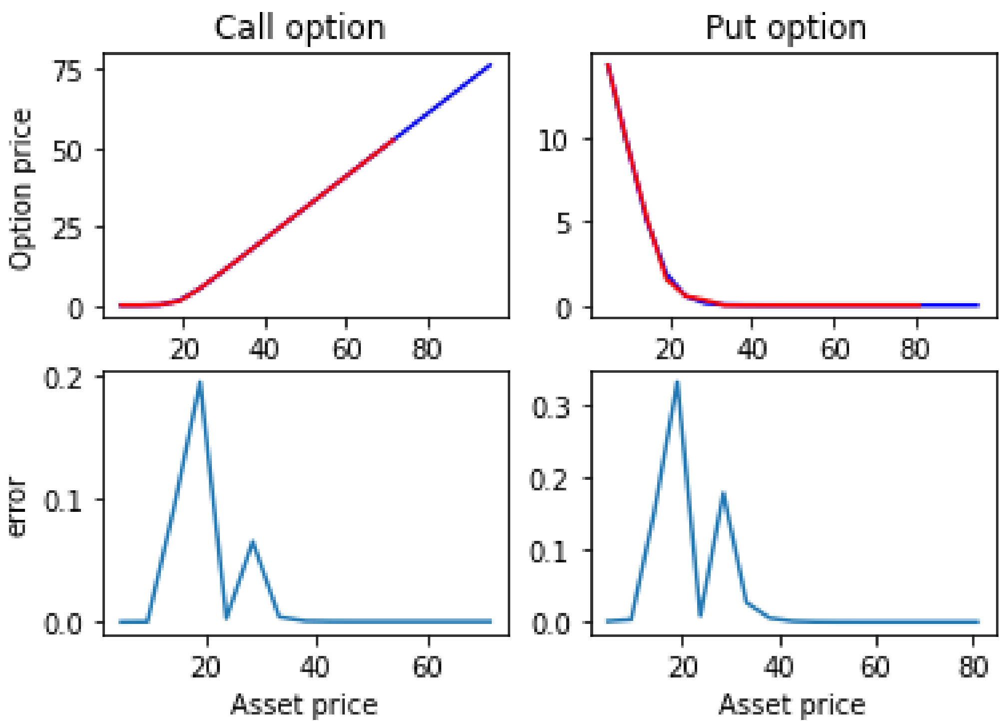


FIGURE 1. Call and put option prices.

Example 1. *The call-and-put option price model with strike price 20, maturity times 1, interest rate 0.05, and volatility 0.25 for both options will be as figure 1.*

As shown in figure 1, the first row shows option prices; the blue line is the true value obtained from Eq. 3, and the red line is the result of our method. As you can see, you can hardly recognize the difference between the red line and blue line, and our method results show the very high efficiency of our method. For a better view, we plotted the error chart of each price figure below it. In a documented way, we can compare the method of this article and the true amount, which, according to the amount of options prices, the amount of errors is very desirable in each step of asset prices.

The call option graph is not complete, and its full chart is shown in figure 2.

As can be seen, the solution is unstable with respect to S. For this problem, without losing the generality of the problem, we increase the price range or S_{max} enough to meet the needs of the problem and avoid the need for unstable points. We will study it in detail in the next research and the next series of this article.

At the end of this paper, to better understand this method's algorithm, put option Python code puts in the appendix as an example.

3|Credit risk transfer bond (CRTB)

Credit risk transfer bond (CRTB)

As mentioned, there are many ways to cover credit risk. In this section, we intend to provide a credit risk-covering mechanism. In the following section, the behavior of CRTB will be studied, and its pricing models will be extracted according to its mechanism.

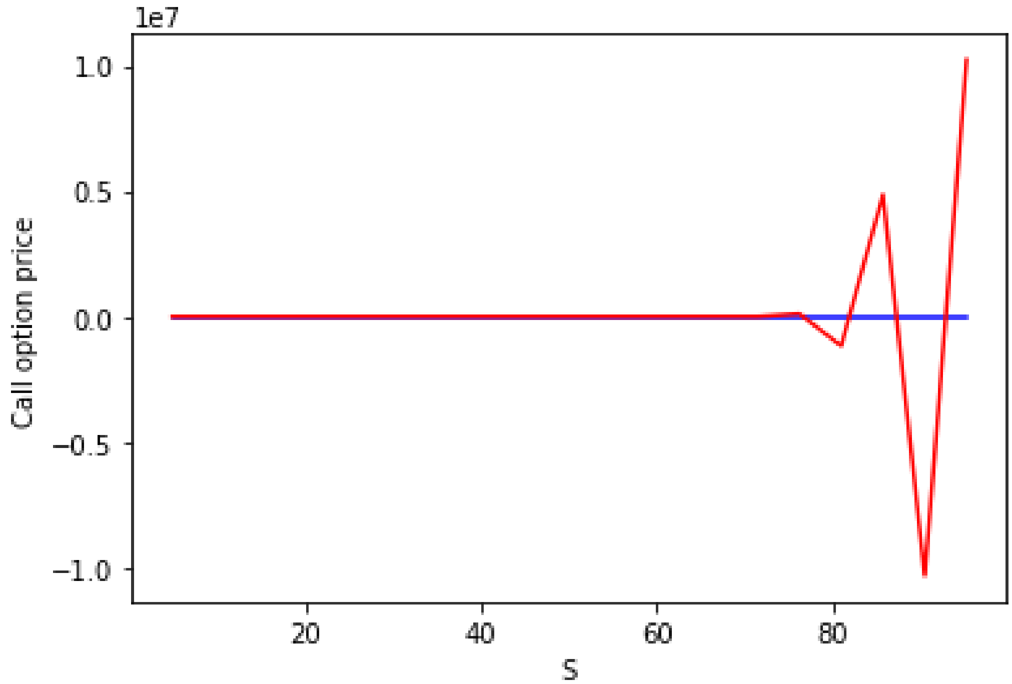
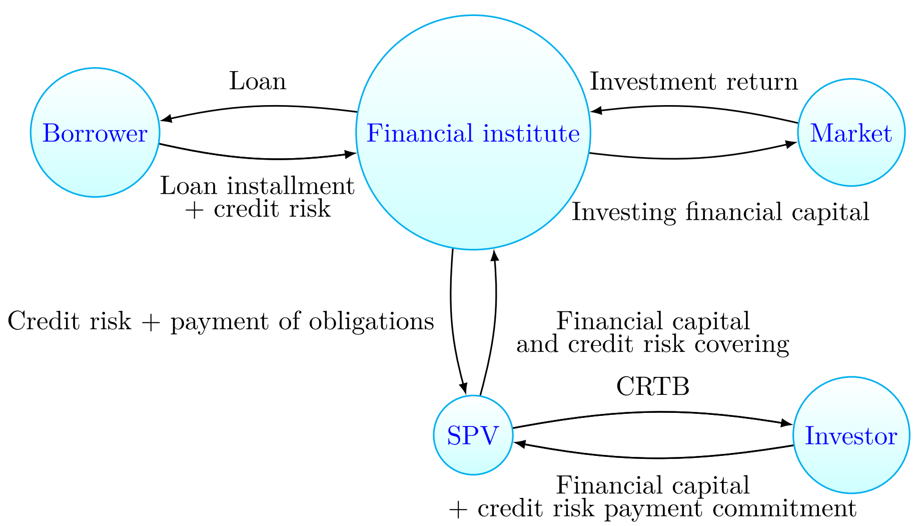


Figure 2. instability of call option model

3.1|CRTB mechanism

CRTB is a type of insurance bond that can protect financial institutions (for example, banks) from credit risk. The mechanism of these bonds is similar to catastrophic insurance bonds. The financial institution enters into the transaction as a risk seller and the investor as a risk buyer, and the risk is transferred from the financial institution to the investor. The chart of the mechanism of these securities is generally as follows:



After entering into a contract with the second party (Borrower) to the contract, the financial institution examines his/her credit risk and estimates the amount of the probability losses, and informs the Special Purpose Vehicle (SPV) about it. Given the amount of damage that the financial institution may face, the SPV starts issuing CRTBs and sells them to risk buyers or investors. The investor undertakes that in case of bankruptcy of the second party to the contract of the financial institution, the loss will be reduced from the principal or return of this bond. The financial institution is also obliged to pay returns according to the type of bond.

Advantages of CRBT.

- The maximum amount of the investor's loss is equal to the principal and the return of the bond, but in the swap, the risk buyer is obligated to pay all the damages caused by the bankruptcy.
- By using bonds, potential losses can be divided among a large number of people, and this is done by issuing more bonds and this work is much easier than swaps.
- Since the loss of each bond can be reduced by issuing it more, more customers can be attracted.
- The bond can attract the potential financial flow and it has a lot of advantages for the financial institutes.

3.2|CRTB modeling

In this section, we intend to use mathematical instruments to reach a price model for bonds. Hence we assume, this bond is a zero-coupon and fix-rate and only its price at any time depends on the credit risk of a borrower. We use the Klein modeling to better understand credit risk. To account for credit risk, Klein calculates the value of the borrower's corporate assets that we show with $K(t)$, which follows a random process as follows:

$$\frac{dK}{K} = rdt + \sigma dw_t \tag{18}$$

where r is interest rate, σ is assets value volatility and w_t is a standard Wiener process. Now suppose D is the sum of the liability of this company that includes loan, and according to these hypotheses, if at maturity $K < D$ this company is in bankruptcy. In this case, the borrower will not be able to repay the loan and the value of the bond must be deducted by $D - K$.

Now, we assume that $B(K, t)$ is the bond value at time t and the value of the borrower's assets are K. The value changes of this bond using the Ito lemma are shown as follows [16, 17]:

$$dB = \underbrace{(B_t + rKB_K + 0.5\sigma^2 K^2 B_{KK})}_{\varphi_B} dt + \underbrace{(\sigma K B_K)}_{\Delta_B} dw_t. \tag{19}$$

According to the above symbols, we have

$$dB = \varphi_B dt + \Delta_B dw_t \tag{20}$$

Consider another bond similar to B called D, whose changes, such as B, are considered as follows:

$$dD = \varphi_D dt + \Delta_D dw_t \tag{21}$$

Now consider a portfolio containing the previous two bonds as Π.

$$\Pi = x_1 B + x_2 D \tag{22}$$

where x_1 is the volume of B and x_2 is the volume of D in the portfolio. Suppose this portfolio is risk-free, so

$$E[d\Pi] = r\Pi dt \tag{23}$$

where E is Expected value in a hypothetical probabilistic space. Considering $x_1 = \Delta_D$ and $x_2 = -\Delta_B$ we will easily reach the following equation:

$$\frac{\varphi_B - rB}{\Delta_B} = \frac{\varphi_D - rD}{\Delta_D} \tag{24}$$

This ratio is called the market price of risk and is denoted by q.

By placing the expressions instead of the symbols, we will reach the following PDE:

$$B_t + (rK - q\sigma K)B_K + 0.5\sigma^2 K^2 B_{KK} - rB = 0 \tag{25}$$

For the final condition, we assume that the borrower's condition is such that he/she may not be able to repay B^* of the total loan in maturity, and SPV issues this amount of bonds. Hence, the pay off function is as follows:

$$B(K,T) = \max(B^* - (D-K), 0)I_{K<D} + B^* I_{K \geq D}. \tag{26}$$

We also have the boundary conditions as follows:

$$\begin{array}{l} B(0,t) = 0 \\ \lim\limits_{K \to \infty} B(K,t) = B^* e^{-r(T-t)} \end{array} \tag{27}$$

In fact, the above functions mean that if the value of the assets associated with the loan of the borrower company is zero, no one buys these bonds, so the bond price is zero. On the other hand, if the value of the company's assets is infinite, the borrower able to repay the loan and this bond act like a normal zero-coupon and fixed-rate bond.

4|Solving method

In this section, we will solve PDE 25 using the method described in section . Consider the changes interval of K as $[0, K_{max}]$, and also the value of K_{max} to be large enough that the instability conditions are far from the value under study. Divide the interval $[0, K_{max}]$ into $n+2$ and consider as follows:

$$\begin{array}{l} K_0 = 0, K_1, ..., K_n, K_{n+1} = K_{\max} \\ dK = K_{i+1} - K_i, \quad i \in \{0, 1, ..., n\} \end{array} \tag{28}$$

By changing the variable $\tau = T - t$, we will get the following inhomogeneous linear system:

$$\dot{B} = AB + C(\tau), \tag{29}$$

where $B := [B_1(tau), \dots, B_n(tau)]^{Transpose}]$, $\dot{B}$ is the derivative respect to time, A is a tridiagonal matrix of derivatives and C is the boundary condition as follows:

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_3 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix}, \quad C(\tau) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_n B^* e^{-r(\tau)} \end{bmatrix} \tag{30}$$

where

$$\begin{array}{l} a_i = \frac{0.5\sigma^2 K_i^2}{dK^2} - (\mathrm{r}\frac{K_i}{2dK} - q\sigma K_i) \\ b_i = -\frac{\sigma^2 K_i^2}{dK^2} - r \\ c_i = \frac{0.5\sigma^2 K_i^2}{dK^2} + (\mathrm{r}\frac{K_i}{2dK} - q\sigma K_i) \end{array} \tag{31}$$

Now using the theorem 1, we have

$$B(\tau) = e^{A\tau} B(K,0) + \int_0^\tau e^{A\tau} e^{-Az} C(z) dz \tag{32}$$

For obtaining the result we use the Taylor series and composite Simpson integral as below:

$$\begin{array}{l} B(\tau) \simeq I + A\tau + \frac{A^2\tau^2}{2!} + \frac{A^3\tau^3}{3!} + \frac{h}{3}(f(z_1) + 4f(z_2) + 2f(z_3) + ... \\ \quad +2f(z_{m-2}) + 4f(z_{m-1}) + f(z_m)) + O, \end{array} \tag{33}$$

where O is the error of this approximation and $f(z_i)$ is as follows:

$$\begin{array}{l} f(z_i) = e^{A\tau} e^{-Az_i} C(z_i) \\ z_1 = 0, z_2, ..., z_m = \tau \\ h = z_{i+1} - z_i, \quad i \in \{1, ..., m-1\} \end{array} \tag{34}$$

Numerical results. In this section, we will run numerical results with Python software. To run a numerical example, we use the data of the Kim and Koo article [12]. For this purpose, we assume $\sigma = 0.125$, $r = 0.05$, $B^* = 30$, $K_{max} = 400$, $D = 80$, $n = 25$, and $m = 21$ so its graph will be as follows:

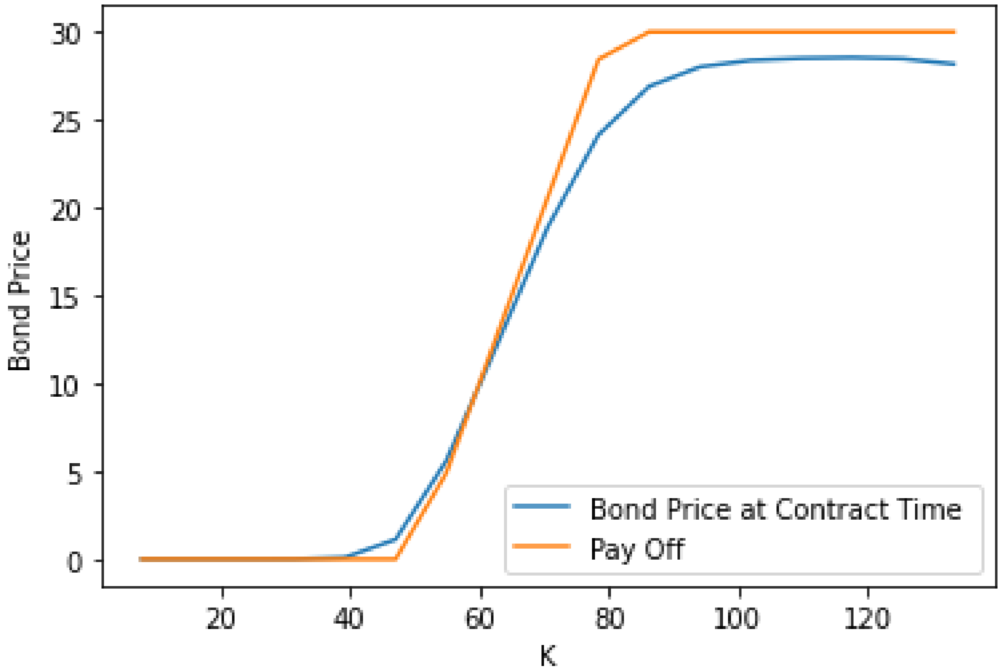


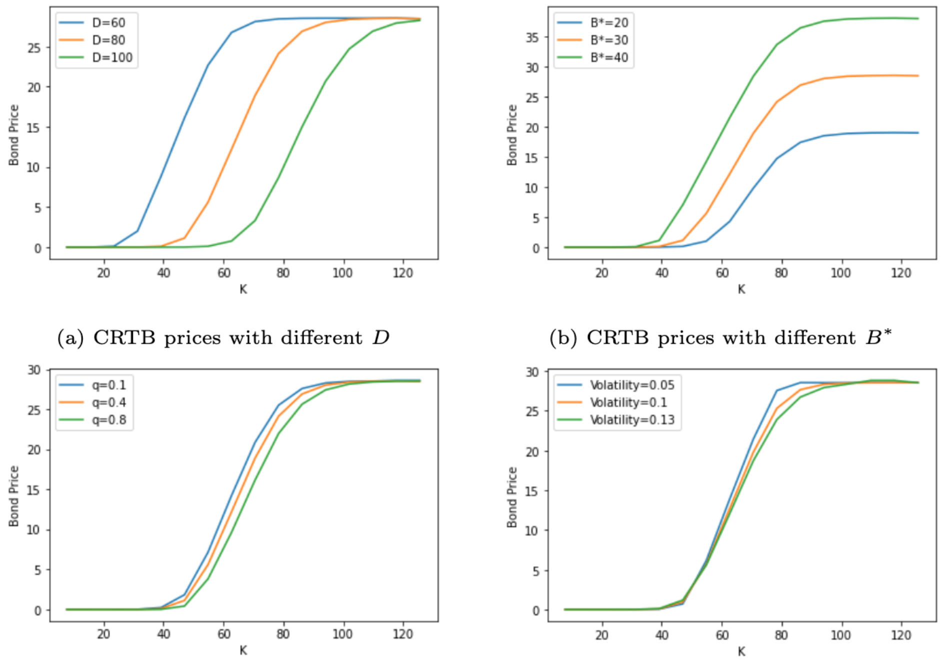
FIGURE 3. CRTB price

As can be seen from the diagram, the blue line indicates the bond price at the time of the contract, and with the increase of K, the bond value increases and behaves like a normal bond from about $K = 90$ onwards. In addition, the figure 4 is provided for further details.

In 4.(a), three graphs is seen for different values of D. As it can be, with the increase of D, the bond price decreases, but eventually, everyone will converge to the same price. The figure 4.(b) shows different graphs for different B^*. As can be seen, the general structure of all three graphs is similar to each other and only the heights of the graphs are different. Also in the chart 4.(c), we see changes in the chart to different amounts of market price risk. According to the chart, with the increase of q, the price of this bond decreases and has an inverse relationship. Another point that can be seen from this chart is that the market price of risk effect becomes more tangible for cases where the prices change fast in a neighborhood of a point, or in other words, in parts of the chart where the derivative value is close to zero, the change of q has no effects. The chart 4.(c) shows changes in bond prices relative to volatility and, unlike other cases, does not follow a strictly decreasing or increasing trend. When the value of K is approximately less than 55, increasing Volatility increases the bond price, but when $K > 55$, the bond price decreases with increasing Volatility.

5|Conclusion

The Black Scholes PDE is one of the PDEs found in many financial studies. Sometimes, the situation is such that a close form can not be found, and other methods, such as numerical methods, are used to solve it. This paper presents a new method for solving this kind of PDE. An efficient solution technique has been provided using dynamic systems. In detail, the price variable is discretized using semi-discretization, and an inhomogeneous linear dynamic system is obtained. This inhomogeneous linear dynamics system is solved using a theorem, to evaluate the efficiency, this method has tested on the call and put option, and we found that this method is very efficient and suitable. In examining the stability of this method, we find that the solution is unstable according to the price variable, to solve it, we expand the interval of asset price changes to meet the needs of the problem and eliminate the unstable points. A special type of credit risk derivative called credit risk transfer bond is introduced, which is used to transfer credit risk from a financial institution to a risk buyer or investor. After presenting the mechanism of this bond, its price model, which is a PDE similar to Black Scholes PDE, was



(a) CRTB prices with different D

(b) CRTB prices with different B^*

(c) CRTB prices with different volatility q

(d) CRTB prices with different volatility

FIGURE 4. CRTB prices with different parameters values

obtained using mathematical models, and it was solved by the mentioned dynamics system method. This bond price depends on the credit risk of the second party of a contract, and the credit risk works like asset price at the option pricing in the dynamic system technique. Eventually, the results are shown by PYTHON software. This study can be very extensive and effective in solving other PDEs. Also, the behavior and stability of this method can be further studied, which we will examine in detail in a future study.

Acknowledgments

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their invaluable comments and constructive feedback, which significantly contributed to the enhancement of this paper.

Author Contribution

M. Azimi: methodology, software, and editing. M. Dabaghian Amiri: conceptualization and writing. S.A.Waloo: writing and editing. All authors have read and agreed to the published version of the manuscript.

Funding

The authors declare that no external funding or support was received for the research presented in this paper, including administrative, technical, or in-kind contributions. Data Availability All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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Appendix

```
# Put option with with strike price 20, maturity times 1,
# interest rate 0.05, volatility 0.25 and the number of discretization 17

import numpy as np
import matplotlib.pyplot as plt

#Matrix exponential func.
def expA(A,t):
    n=A.shape
    n=n[0]
    S=0
    PA=np.eye(n)
    T=1
    for i in range(3):
        T=T*t
        PA=np.matmul(PA,A)
        S=S+PA*T/np.prod(range(1,i+1+1))
    Z=np.eye(n)+S
    return Z

#Integral part
def Integral(a,r,k,t,A):
    S,n=0,10
    m=A.shape
    m=m[0]
    BC=np.zeros((m,1)) #Boundary condition
    T=np.linspace(0,t,n)
    h=T[1]-T[0]
    for i in range(n):
        BC[0]=a*k*np.exp(-r*T[i])
        if i==0 or i==n-1:
            S=S+np.matmul(expA(A,t-T[i]),BC)
        else:
            S=S+2*np.matmul(expA(A,t-T[i]),BC)
    return h/2*S

def putOption(a,IC,r,k,t,A):
    return np.matmul(expA(A,t),IC)+Integral(a,r,k,t,A)

#Discretization
s=np.linspace(0,100,17)
ds=s[1]-s[0]

#Derivative matrix
a=0.5*0.25*0.25*s*s/(ds*ds)-0.05*s/(2*ds)
b=-0.25*0.25*s*s/(ds*ds)-0.05
c=0.5*0.25*0.25*s*s/(ds*ds)+0.05*s/(2*ds)
A=np.diag(b[1:16])+np.diag(a[2:16],k=-1)+np.diag(c[1:15],k=1)

#Initial condition
IC=20-s
IC=np.max(np.hstack((IC.reshape((17,1)),np.zeros((17,1)))),1)
IC=IC.reshape((17,1))

#Pricing func.
PU=putOption(a[1],IC[1:16],0.05,20,1,A)

#Plotting
plt.plot(s[1:-1],PU,'r')
plt.ylabel('Put option price')
plt.xlabel('S')
plt.show()
```