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# Nonlinear Signal Decomposition with Bilinear Hilbert Transform: A Framework for Analytical Decision-Making Applications

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**Abstract**

This note emphasizes a point of view on the Bilinear Hilbert Transform (BHT) in the Bedrosian identity, which is originally based on the conventional Hilbert Transform in the theory of analytic signals. We show that the generalized sinc functions from Möbius play a crucial role. We have demonstrated some properties of BHT: Riesz representation theorem, boundedness, and Bedrosian identity for BHT. The generalized sinc function is a special solution of the Bedrosian identity. After that, we consider some of the nonlinear sinc functions, which are also the solutions of the Bedrosian identity. Lastly, the nonlinear sinc function system is orthonormal. In addition, we propose a framework for analytical decision-making applications that leverages the properties of BHT. incorporating this framework into existing real-world systems further enhances the adaptability and responsiveness of decision-making models, positioning the BHT as a critical tool for optimizing processes in dynamic, data-rich environments.

**Keywords:**Bilinear Hilbert transform,Bedrosian identity,Nonlinear Fourier atoms.

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# 1|The Introduction of Bilinear Hilbert Transform

Denote the translation and modulation of a signal $f$ by

$$T_a f(t) = f(t-a), \quad M_b f(t) = e^{ibt} f(t), \quad t \in \mathbb{R}.$$

The vector-valued Hilbert transform and the bilinear Hilbert transform(BHT)[4, 6] in $\mathbb{R}$ are defined as follows

$$Hf(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(t-x)\frac{1}{x}dx$$

and

$$H(f,g)(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(t-x)g(t+x)\frac{dx}{x}.$$

**Theorem 1.** *BHT is bilinear transform.*

*Proof.*

$$\begin{aligned} H(\alpha f+\beta g,h)(t) &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}[\alpha f(t-x)+\beta g(t-x)]h(t+x)\frac{dx}{x} \\ &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\alpha f(t-x)h(t+x)\frac{dx}{x}+\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\beta g(t-x)h(t+x)\frac{dx}{x} \\ &= \alpha\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(t-x)h(t+x)\frac{dx}{x}+\beta\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} g(t-x)h(t+x)\frac{dx}{x} \\ &= \alpha H(f,h)(t)+\beta H(g,h)(t). \end{aligned}$$

Let's verify $H(f,g)(t) = H(g,f)(t)$

$$\begin{aligned} H(f,g)(t) &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(t-x)g(t+x)\frac{dx}{x} \\ &= -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(x)g(2t-x)\frac{dx}{t-x} \\ H(g,f)(t) &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} g(t-x)f(t+x)\frac{dx}{x} \\ &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(x)g(2t-x)\frac{dx}{x-t} \end{aligned}$$

$H(f,g)(t) = H(g,f)(t)$,BHT is bilinear transform. □

We give an alternative characterization of BHT by the Fouerier transform[7, 13]. Fix a Schwartz function $\psi$ on $\mathbb{R}$ ,let $\psi = f(x-t)g(x+t)$.Thereinto $t$ is fixed and $x$ is variable,then

$$\begin{aligned} <\hat{w_0},\psi> = <w_0,\hat{\psi}> &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\hat{\psi}(x)\frac{dx}{x} \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\frac{1}{\varepsilon}\geq|x|\geq\varepsilon}\int_R \psi(\xi)e^{-2\pi ix\xi}d\xi\frac{dx}{x} \\ &= \lim_{\varepsilon\to 0}\int_R\psi(\xi)[\frac{1}{\pi}\int_{\frac{1}{\varepsilon}\geq|x|\geq\varepsilon}e^{-2\pi ix\xi}\frac{dx}{x}]d\xi \\ &= \lim_{\varepsilon\to 0}\int_R\psi(\xi)[\frac{-i}{\pi}\int_{\frac{1}{\varepsilon}\geq|x|\geq\varepsilon}sin(2\pi x\xi)\frac{dx}{x}]d\xi \\ &= \lim_{\varepsilon\to 0}\int_R\psi(\xi)[\frac{-i}{\pi}sgn(\xi)\int_{\frac{1}{2\pi\varepsilon}\geq|x|\geq\frac{\varepsilon}{2\pi}}sin(x|\xi|)\frac{dx}{x}]d\xi \end{aligned}$$

The Lebesgue dominated convergence theorem allows the passage of the limit inside the integral.We obtain that

$$<\hat{w_0},\psi> = \int_R\psi(\xi)(-isgn(\xi))d\xi$$

This implies that

$$\widehat{w_0}\,(x) = -isgn(x)$$

We use identity$< \hat{w_0}, \psi >= \int_R \psi(\xi)(-isgn(\xi))d\xi$ to write

$$H(f,g)(t) = \left(f\,(x-\widehat{t)\,g\,(}x+t)\,(-isgn(x))\right)^{\vee}(t)$$

From the above we can see that the Riesz representation theorem applies to BHT.
For all $1 < p < 2$ ,we give the $L^P$ boundedness of BHT[2, 3, 5].

**Lemma 2.** *For BHT given by*

$$H\,(f,g) = \frac{1}{\pi}p.v.\int_R \frac{f\,(t-x)\,g\,(t+x)}{x}dx$$

*the inequality*

$$\|H\,(f,g)\|_{p_3} \leqslant K_{p_1p_2}\|f\|_{p_1}\|g\|_{p_2}$$

*exsits, provided* $2 < p_1, p_2 < \infty$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $1 < p_3 < 2$.

The following lemma establishes a relationship betweeen Hilbert transform and BHT by modulation operator.

**Lemma 3.** *Set* $e_\xi(\cdot) = e^{i<\xi,\cdot>}$. *For any* $\xi \in \mathbb{R}$, *we have*

$$H(M_\xi f)(t) = -M_{2\xi}H(f, e_{-\xi})(t), t \in \mathbb{R}. \tag{1}$$

*Proof.* By the definition of BHT and a change of variable in the integral, we get

$$\begin{aligned} H(f, e_\xi)(t) &= \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(t-y)e_\xi(t+y)\frac{dy}{y} \\ &= -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(y)e_\xi(2t-y)\frac{dy}{t-y} \\ &= -e^{i<\xi,2t>}\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} f(y)e^{-i<\xi,y>}\frac{dy}{t-y}. \end{aligned}$$

By recalling the definition of modulation and Hilbert transform, we obtain

$$H(f, e_\xi)(t) = -e^{i<2\xi,t>}H(M_{-\xi}f)(t),$$

that is

$$H(M_{-\xi}f)(t) = -M_{-2\xi}H(f, e_\xi)(t).$$

We therefore conclude that this is an alternative form of (1) by setting $\xi$ instead of $-\xi$. The proof of this lemma is finished. □

The theorem below states that Bedrosian identity[15, 16] can be formulated by BHT instead of Hilbert transform.

**Theorem 4.** *Suppose that* $g \in L^2[(0, 2\pi)]$ *is* $2\pi$*-periodic function of real variable. Then the Bedrosian identity*

$$H(fg)(t) = f(t)Hg(t), \quad t \in \mathbb{R} \tag{2}$$

*is equivalent to*

$$H(f, g(2t-\cdot))(t) = f(t)H(1, g(2t-\cdot))(t), \quad t \in \mathbb{R}. \tag{3}$$

*Proof.* Since $g \in L^2[(0,2\pi)]$, $g$ has a convergent Fourier expansion $g(t) = \sum_{k\in\mathbb{Z}} c_k(g)e^{ikt}$. One hand, by linearity of Hilbert transform, we have

$$\begin{aligned} H(fg)(t) &= H\left(f(\cdot)\sum_{k\in\mathbb{Z}} c_k(g)e^{i<k,\cdot>}\right)(t) \\ &= \sum_{k\in\mathbb{Z}} c_k(g)H\left(f(\cdot)e^{i<k,\cdot>}\right)(t) \\ &= \sum_{k\in\mathbb{Z}} c_k(g)H(M_kf)(t). \end{aligned}$$

Using the equation (1) and the linearity of BHT implies

$$\begin{aligned} H(fg)(t) &= -\sum_{k\in\mathbb{Z}} c_k(g)M_{2k}H(f,e_{-k})(t) \\ &= -\sum_{k\in\mathbb{Z}} c_k(g)e^{i2kt}H(f,e_{-k})(t) \\ &= -H\left(f,\sum_{k\in\mathbb{Z}} c_k(g)e^{i2kt}e_{-k}\right)(t) \\ &= -H\left(f,\sum_{k\in\mathbb{Z}} c_k(g)e^{i<k,2t-\cdot>}\right)(t) \\ &= -H\left(f,g(2t-\cdot)\right)(t). \end{aligned}$$

One the other hand, by the similar technique, we have that

$$f(t)Hg(t) = -f(t)H(1,g(2t-\cdot))(t).$$

We therefore conclude that the two equations are equivalent each other. The proof of this theorem is completed. □

# 2|Bedrosian identity for nonlinear Fourier atoms

This section is focusing to investigate the Bedrosian identity of BHT type in the case that $g$ is nonlinear Fourier atom[8], that is, we want to study the solutions of the following equation

$$H\left(\rho,\cos\theta_a(2t-\cdot)\right)(t) = \rho(t)H\left(1,\cos\theta_a(2t-\cdot)\right)(t), \quad t\in\mathbb{R}. \tag{4}$$

The following theorem shows that the generalized Sinc function is a special solution of above equation.

**Theorem 5.** *The generalized Sinc function* $Sinc_a(t) := p_a(t)\frac{\sin t}{t} = \frac{\sin\theta_a(t)}{t}$ *satisfies the equation (4), that is,*

$$H(Sinc_a(\cdot),\cos\theta_a(2t-\cdot))(t) = Sinc_a(t)H(1,\cos\theta_a(2t-\cdot))(t) = \frac{\sin^2\theta_a(t)}{t}, \quad t\in\mathbb{R}. \tag{5}$$

To prove this theorem, we need some preparations. The first fact is that the so called one-sided ladder shape filter $H_a^+(t) = a^{\lfloor |t| \rfloor}\chi_{\mathbb{R}}(t)$ and the function $r(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{it}}\frac{1-e^{it}}{-it}$ form a pair of Fourier transform, that is

**Lemma 6.**

$$(H_a^+)^\wedge(\xi) = r(-\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{-i\xi}}\frac{1-e^{-i\xi}}{i\xi}$$

*and*

$$\hat{r}(\xi) = \left(\frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{i\cdot}}\frac{1-e^{i\cdot}}{-i\cdot}\right)^\wedge(\xi) = H_a^+(\xi).$$

*Proof.* The detail of calculation is below

$$
\begin{aligned}
(H_a^+)^\wedge(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H_a^+(t)e^{-i\xi t}dt \\
&= \frac{1-a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty} a^{k-1}\chi_{[0,k)}(t)e^{-i\xi t}dt \\
&= \frac{1-a}{\sqrt{2\pi}}\sum_{k=1}^{\infty} a^{k-1}\int_0^k e^{-i\xi t}dt = \frac{1-a}{\sqrt{2\pi}}\sum_{k=1}^{\infty} a^{k-1}\frac{1-e^{-ik\xi}}{i\xi} \\
&= \frac{1-a}{\sqrt{2\pi}}\left(\frac{1}{1-a}-\frac{e^{-i\xi}}{1-ae^{-i\xi}}\right)/(i\xi) = \frac{1}{\sqrt{2\pi}}\frac{1-ae^{-i\xi}-(1-a)e^{-i\xi}}{i\xi(1-ae^{-i\xi})} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{-i\xi}}\frac{1-e^{-i\xi}}{i\xi}.
\end{aligned}
$$

The proof of this lemma is finished. □

The second fact is the representation of $\cos\theta_a(t)$ and $\sin\theta_a(t)$.

**Lemma 7.** *For real number $a \in (-1,1)$, the following equations hold*

$$
\begin{aligned}
\sin\theta_a(t) &= p_a(t)\sin t = \frac{(1-a^2)\sin t}{1-2a\cos t + a^2}, \quad t \in \mathbb{R}, \\
\cos\theta_a(t) &= \frac{(1+a^2)\cos t - 2a}{1-2a\cos t + a^2}, \quad t \in \mathbb{R}.
\end{aligned}
$$

*Proof.* The boundary on the unit circle of Möbius transform can conclude the two formulae. The detail is as follows

$$
\begin{aligned}
\tau_a(z)|_{z=e^{it}} &= \frac{e^{it}-a}{1-ae^{it}} = \frac{(e^{it}-a)(1-ae^{-it})}{1-2a\cos t + a^2} \\
&= \frac{e^{it}-2a+a^2e^{-it}}{1-2a\cos t + a^2} = \frac{(1+a^2)\cos t - 2a}{1-2a\cos t + a^2} + i\frac{(1-a^2)\sin t}{1-2a\cos t + a^2}.
\end{aligned}
$$

The proof of this lemma is finished. □

The third is the formula of the Hilbert transform of generalized Sinc function

**Lemma 8.** *The Hilbert transform of $Sinc_a$ is*

$$
H(Sinc_a)(t) = \frac{1+a}{1-a}p_a(t)\frac{1-\cos t}{t} = \frac{1-\cos\theta_a(t)}{t}, \quad t \in \mathbb{R}. \tag{6}
$$

*Proof.* We have shown that the Fourier transform of $r(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{it}}\frac{e^{it}-1}{it}$ is the one-sided ladder shape function $H_a^+(\xi)$. Since $r$ has no negative frequency, it is an analytic signal and therefore the real part and imaginary part form a pair of Hilbert transform

$$
H(\mathrm{Re}r)(t) = \mathrm{Im}r(t), \quad t \in \mathbb{R}.
$$

Note that

$$
\begin{aligned}
r(t) &= \frac{1}{\sqrt{2\pi}}\frac{(1-a)\sin t}{1-2a\cos t + a^2}\frac{1}{t} + i\frac{1}{\sqrt{2\pi}}\frac{(1+a)(1-\cos t)}{1-2a\cos t + a^2}\frac{1}{t} \\
&= \frac{1}{\sqrt{2\pi}(1+a)}p_a(t)\frac{\sin t}{t} + i\frac{1}{\sqrt{2\pi}(1-a)}p_a(t)\frac{1-\cos t}{t}.
\end{aligned}
$$

We get

$$
H\left(p_a(\cdot)\frac{\sin(\cdot)}{\cdot}\right)(t) = \frac{1+a}{1-a}p_a(t)\frac{1-\cos t}{t},
$$

which is equivalent to

$$H\left(\operatorname{Sinc}_a(\cdot)\right)(t)=\frac{1+a}{1-a}p_a(t)\frac{1-\cos t}{t}.$$

We are left to show

$$1-\cos\theta_a(t)=\frac{1+a}{1-a}p_a(t)(1-\cos t),\quad t\in\mathbb{R}.$$

In fact, by the representation of $\cos\theta_a(t)$ (see Lemma 7), we have

$$\begin{aligned}1-\cos\theta_a(t) &= 1-\frac{(1+a^2)\cos t-2a}{1-2a\cos t+a^2}=\frac{(1+a)^2-(1+a)^2\cos t}{1-2a\cos t+a^2}\\ &= \frac{(1+a)^2(1-\cos t)}{1-2a\cos t+a^2}=\frac{1+a}{1-a}p_a(t)(1-\cos t).\end{aligned}$$

The proof of this lemma is completed. □

**Remark 1**: This lemma indicates an interesting formula: pseudo-Bedrosian formula

$$H(p_a(\cdot)\operatorname{Sinc}(\cdot))(t)=\frac{1+a}{1-a}p_a(t)H(\operatorname{Sinc}(\cdot))(t),\quad t\in\mathbb{R}.$$

**Remark 2**: There is an alternative approach to prove this lemma. By using Qian's result[11, 12] $H\cos\theta_a(t)=\sin\theta_a(t)$, we get

$$\begin{aligned}H\left(\operatorname{Sinc}_a(\cdot)\right)(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}\frac{dy}{t-y}\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sin\theta_a(y)\frac{1}{t}(\frac{1}{y}+\frac{1}{t-y})dy\\ &= \frac{1}{t}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}dy+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{t-y}dy\right)\\ &= \frac{1}{t}\left(-H(\sin\theta_a)(0)+H(\sin\theta_a)(t)\right)=\frac{1}{t}\left(-(-\cos\theta_a(0))-\cos\theta_a(t)\right)\\ &= \frac{1-\cos\theta_a(t)}{t}.\end{aligned}$$

The fourth is the Hilbert transform of the function $\frac{\sin(2\theta_a(t))}{2t}$ (different from $\operatorname{sinc}_a(2t)$).

**Lemma 9.** *The following formula holds*

$$H\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t)=\frac{\sin^2\theta_a(t)}{t},\quad t\in\mathbb{R}. \tag{7}$$

*Proof.* Direct calculation leads to

$$\begin{aligned}&H\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t)\\ =&\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(2\theta_a(y))}{2y}\frac{dy}{t-y}\\ =&\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\sin(2\theta_a(y))\frac{1}{t}(\frac{1}{y}+\frac{1}{t-y})dy\\ =&\ \frac{1}{2t}\left(-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(2\theta_a(y))}{0-y}dy+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(2\theta_a(y))}{t-y}dy\right)\\ =&\ \frac{1}{2t}\left(-H(\sin(2\theta_a(\cdot)))(0)+H(\sin(2\theta_a(\cdot)))(t)\right).\end{aligned}$$

Utilizing Qian's result[11, 12] in Blaschke product case with two same parameters $H\cos(2\theta_a(\cdot))(t) = \sin(2\theta_a(t))$ leads to

$$H\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t) = \frac{1}{2t}\left(-(-\cos(2\theta_a(0))) - \cos(2\theta_a(t))\right) = \frac{1-\cos(2\theta_a(t))}{2t} = \frac{\sin^2\theta_a(t)}{t}.$$

This completes the proof of this lemma. □

Now it it time to prove Theorem 5.

**Proof of Theorem 5**

*Proof.* Set $\rho = \mathrm{Sinc}_a$. One hand, the left side of (4) equals to

$$\begin{aligned}\text{Left} &= H\left(\rho, \cos\theta_a(2t-\cdot)\right)(t)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(t-y)}{t-y}\cos\theta_a(2t-(t+y))\frac{dy}{y}\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}\cos\theta_a(y)\frac{dy}{t-y}\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(2\theta_a(y))}{2y}\frac{dy}{t-y} = H\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t).\end{aligned}$$

By (7), we conclude that

$$\text{Left} = \frac{\sin^2\theta_a(t)}{t}, \quad t\in\mathbb{R}.$$

One the other hand, the right side of (4) is

$$\begin{aligned}\text{Right} &= \rho(t)H\left(1, \cos\theta_a(2t-\cdot)\right)(t)\\ &= \frac{\sin\theta_a(t)}{t}\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\theta_a(2t-(t+y))\frac{dy}{y} = \frac{\sin\theta_a(t)}{t}H\cos(\theta_a(\cdot))(t).\end{aligned}$$

Using Qian's result[11, 12] again concludes that

$$\text{Right} = \frac{\sin^2\theta_a(t)}{t}, \quad t\in\mathbb{R}.$$

The proof of this theorem is finished. □

The following theorem offers us a more generalized result.

**Theorem 10.** *For any $l^2$ sequence $c = \{c_k\}$, the function*

$$\rho(t) = \sum_{k\in\mathbb{Z}} c_k\frac{\sin\theta_a(t-2k\pi)}{t-2k\pi} = \sum_{k\in\mathbb{Z}} c_k Sinc_a(t-2k\pi), \quad t\in\mathbb{R}$$

*satisfies the equation (4)*

*Proof.* We first establish an important equation. By (5), i.e. (rewrite it for convenience)

$$H(\mathrm{Sinc}_a(\cdot), \cos\theta_a(2t-\cdot))(t) = \mathrm{Sinc}_a(t)H(1, \cos\theta_a(2t-\cdot))(t) = \frac{\sin^2\theta_a(t)}{t},$$

we know that, for any integer $k$, the following equation holds

$$\begin{aligned}H(\mathrm{Sinc}_a(\cdot), \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi) &= \mathrm{Sinc}_a(t-2k\pi)H(1, \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi)\\ &= \frac{\sin^2\theta_a(t-2k\pi)}{t-2k\pi}.\end{aligned}$$

By the definition of $\rho$, the linearity of BHT and the $2\pi$-periodicity of $\cos\theta_a(\cdot)$, we get

$$\begin{aligned} & H(\rho(\cdot), \cos\theta(2t-\cdot))(t) \\ = & \sum_{k\in\mathbb{Z}} c_k H\left(\mathrm{Sinc}_a(\cdot - 2k\pi), \cos\theta_a(2t-\cdot)\right)(t) \\ = & \sum_{k\in\mathbb{Z}} c_k \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Sinc}_a(t-y-2k\pi)\cos\theta_a(2t-(t+y))\frac{dy}{y} \\ = & \sum_{k\in\mathbb{Z}} c_k \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Sinc}_a((t-2k\pi)-y)\cos\theta_a(t-y)\frac{dy}{y} \\ = & \sum_{k\in\mathbb{Z}} c_k \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Sinc}_a((t-2k\pi)-y)\cos\theta_a(2(t-2k\pi)-((t-2k\pi)+y))\frac{dy}{y} \\ = & \sum_{k\in\mathbb{Z}} c_k H(\mathrm{Sinc}_a(\cdot), \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi). \end{aligned}$$

By using the important equation we established at beginning, we get

$$\begin{aligned} & H(\rho(\cdot), \cos\theta(2t-\cdot))(t) \\ = & \sum_{k\in\mathbb{Z}} c_k \mathrm{Sinc}_a(t-2k\pi) H(1, \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi) \\ = & \sum_{k\in\mathbb{Z}} c_k \mathrm{Sinc}_a(t-2k\pi)\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\theta_a\left(2(t-2k\pi)-((t-2k\pi)+y\right)\frac{dy}{y} \\ = & \sum_{k\in\mathbb{Z}} c_k \mathrm{Sinc}_a(t-2k\pi)\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\theta_a(2t-(t+y))\frac{dy}{y} \\ = & \sum_{k\in\mathbb{Z}} c_k \mathrm{Sinc}_a(t-2k\pi) H(1, \cos\theta_a(t-\cdot))(t) \\ = & \rho(t) H(1, \cos\theta_a(2t-\cdot))(t). \end{aligned}$$

The proof of this theorem is completed. □

# 3|Nonlinear Sinc-function and Bilinear Hilbert transform

**Lemma 11.** *The following equation holds*

$$H\left[(HSinc_a)(\cdot)\cos\theta_a(\cdot)\right](t) = (HSinc_a)(t)\sin\theta_a(t), \quad t\in\mathbb{R}. \tag{8}$$

*Proof.* The fact $H\mathrm{Sinc}_a(t) = \frac{1-\cos\theta_a(t)}{t}$ implies that we can write the equation (8) as

$$H\left(\frac{1-\cos\theta_a(\cdot)}{\cdot}\cos\theta_a(\cdot)\right)(t) = \frac{1-\cos\theta_a(t)}{t}\sin\theta_a(t), \quad t\in\mathbb{R}.$$

Noting that $-H^2$ equals to identity operator, we are left to show the equivalent equation

$$H\left(\frac{1-\cos\theta_a(\cdot)}{\cdot}\sin\theta_a(\cdot)\right)(t) = -\frac{1-\cos\theta_a(t)}{t}\cos\theta_a(t), \quad t\in\mathbb{R}.$$

This can be checked by direct calculation combing with the equations (6) and (7)

$$\begin{aligned} & H\left(\frac{1-\cos\theta_a(\cdot)}{\cdot}\sin\theta_a(\cdot)\right)(t) \\ = & H\mathrm{Sinc}_a(t) - H\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t) \\ = & \frac{1-\cos\theta_a(t)}{t} - \frac{\sin^2\theta_a(t)}{t} \\ = & -\frac{1-\cos\theta_a(t)}{t}\cos\theta_a(t). \end{aligned}$$

This completes the proof of this lemma. □

**Theorem 12.** *The function $\rho(t) = HSinc_a(t)$ is a solution of the equation (4), that is,*

$$H(HSinc_a(\cdot), \cos\theta_a(2t-\cdot))(t) = HSinc_a(t)H(1, \cos\theta_a(2t-\cdot))(t), \quad t \in \mathbb{R}. \tag{9}$$

*Proof.* By the definition of BHT and a suitable variable changing, we get

$$\begin{aligned} H(H\text{Sinc}_a(\cdot), \cos\theta_a(2t-\cdot))(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\theta_a(t-y)}{t-y} \cos\theta_a(t-y)\frac{dy}{y} \\ &= H\left[(H\text{Sinc}_a)(\cdot)\cos\theta_a(\cdot)\right](t). \end{aligned}$$

Using (8) and the identity $H(1, \cos\theta_a(2t-\cdot))(t) = \sin\theta_a(t)$ shows that

$$H(H\text{Sinc}_a(\cdot), \cos\theta_a(2t-\cdot))(t) = (H\text{Sinc}_a(t))\sin\theta_a(t) = H\text{Sinc}_a(t)H(1, \cos\theta_a(2t-\cdot))(t).$$

The proof of this theorem is completed. □

**Theorem 13.** *For any $l^2$ sequence $d = \{d_k\}$, the function*

$$\rho(t) = \sum_{k\in\mathbb{Z}} d_k \frac{1-\cos\theta_a(t-2k\pi)}{t-2k\pi} = \sum_{k\in\mathbb{Z}} d_k (HSinc_a)(t-2k\pi), \quad t \in \mathbb{R}$$

*satisfies the equation (4)*

*Proof.* By (9), we first claim that, for any integer $k$,

$$\begin{aligned} & H(H\text{Sinc}_a(\cdot), \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi) \\ = \;& H\text{Sinc}_a(t-2k\pi)H(1, \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi), \quad t \in \mathbb{R}. \end{aligned}$$

By the definition of $\rho$, the linearity of BHT and the $2\pi$-periodicity of $\cos\theta_a(\cdot)$, we get

$$\begin{aligned} & H(\rho(\cdot), \cos\theta(2t-\cdot))(t) \\ = \;& \sum_{k\in\mathbb{Z}} d_k H\left((H\text{Sinc}_a)(\cdot-2k\pi), \cos\theta_a(2t-\cdot)\right)(t) \\ = \;& \sum_{k\in\mathbb{Z}} d_k \frac{1}{\pi}\int_{-\infty}^{\infty} (H\text{Sinc}_a)(t-y-2k\pi)\cos\theta_a(2t-(t+y))\frac{dy}{y} \\ = \;& \sum_{k\in\mathbb{Z}} d_k \frac{1}{\pi}\int_{-\infty}^{\infty} (H\text{Sinc}_a)((t-2k\pi)-y)\cos\theta_a(t-y)\frac{dy}{y} \\ = \;& \sum_{k\in\mathbb{Z}} d_k \frac{1}{\pi}\int_{-\infty}^{\infty} (H\text{Sinc}_a)((t-2k\pi)-y)\cos\theta_a(2(t-2k\pi)-((t-2k\pi)+y))\frac{dy}{y} \\ = \;& \sum_{k\in\mathbb{Z}} d_k H((H\text{Sinc}_a)(\cdot), \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & H(\rho(\cdot), \cos\theta(2t-\cdot))(t) \\ = \;& \sum_{k\in\mathbb{Z}} d_k (H\text{Sinc}_a)(t-2k\pi)H(1, \cos\theta_a(2(t-2k\pi)-\cdot))(t-2k\pi) \\ = \;& \sum_{k\in\mathbb{Z}} d_k (H\text{Sinc}_a)(t-2k\pi)H(1, \cos\theta_a(2t-\cdot))(t) \\ = \;& \rho(t)H(1, \cos\theta_a(2t-\cdot))(t). \end{aligned}$$

This completes the proof of this theorem. □

**Theorem 14.** *For any sequence pairs c and d in $l^2(\mathbb{Z})$, the function*

$$\rho(t) = \sum_{k\in\mathbb{Z}} c_k Sinc_a(t-2k\pi) + \sum_{k\in\mathbb{Z}} d_k (HSinc_a)(t-2k\pi)$$

*satisfies the equation (4).*

*Proof.* This is a direct conclusion of previous two theorems by utilizing the linearity of bilinear Hilbert transform. □

# 4|Necessity

**Lemma 15.** *For any real number a with $|a|<1$, the following identity holds*

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1} min\{l,m\} = \frac{1}{(1-a)^3(1+a)}.$$

*Proof.*

$$\begin{aligned} & \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1} min\{l,m\} \\ = \quad & (\sum_{m\geq l} + \sum_{m<l}) a^{l-1}a^{m-1} min\{l,m\} = \sum_{m=1}^{\infty}\sum_{l=1}^{m} a^{l-1}\,a^{m-1}\,l + \sum_{l=2}^{\infty}\sum_{m=1}^{l-1} a^{m-1}\,a^{l-1}\,m \end{aligned}$$

Note that

$$\sum_{l=1}^{m} la^{l-1} = \frac{d}{da}(\sum_{l=1}^{m} a^l) = \frac{d}{da}(\frac{a-a^{m+1}}{1-a}) = \frac{1-(m+1)a^m + ma^{m+1}}{(1-a)^2}$$

and

$$\sum_{m=1}^{l-1} ma^{m-1} = \frac{d}{da}(\sum_{m=1}^{l-1} a^m) = \frac{d}{da}(\frac{a-a^l}{1-a}) = \frac{1-la^{l-1}+(l-1)a^l}{(1-a)^2}.$$

Thus

$$
\begin{aligned}
& \sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1}min\{l,m\} \\
= & \sum_{m=1}^{\infty} a^{m-1}\frac{1-(m+1)a^m+ma^{m+1}}{(1-a)^2} + \sum_{l=2}^{\infty} a^{l-1}\frac{1-la^{l-1}+(l-1)a^l}{(1-a)^2} \\
= & \sum_{m=1}^{\infty} a^{m-1}\frac{1-(m+1)a^m+ma^{m+1}}{(1-a)^2} + \sum_{m=1}^{\infty} a^{m}\frac{1-(m+1)a^m+ma^{m+1}}{(1-a)^2} \\
= & \frac{1}{(1-a)^2}\sum_{m=1}^{\infty}\left(a^{m-1}-(m+1)a^{2m-1}+ma^{2m}+a^m-(m+1)a^{2m}+ma^{2m+1}\right) \\
= & \frac{1}{(1-a)^2}\sum_{m=1}^{\infty}\left(a^{m-1}+a^m-a^{2m}\right) + \frac{1}{(1-a)^2}\sum_{m=1}^{\infty}\left(ma^{2m+1}-(m+1)a^{2m-1}\right) \\
= & \frac{1}{(1-a)^2}\sum_{m=1}^{\infty}\left(a^{m-1}+a^m-a^{2m}\right) + \frac{1}{(1-a)^2}\sum_{m=1}^{\infty} ma^{2m+1} - \frac{1}{(1-a)^2}\sum_{m=0}^{\infty}(m+2)a^{2m+1} \\
= & \frac{1}{(1-a)^2}\sum_{m=1}^{\infty}\left(a^{m-1}+a^m-a^{2m}\right) - \frac{2a}{(1-a)^2} - \frac{2}{(1-a)^2}\sum_{m=1}^{\infty} a^{2m+1} \\
= & \frac{1}{(1-a)^2}\left(\frac{1}{1-a}+\frac{a}{1-a}-\frac{a^2}{1-a^2}\right) - \frac{2a}{(1-a)^2} - \frac{2}{(1-a)^2}\frac{a^3}{1-a^2} \\
= & \frac{1}{(1-a)^3(1+a)}\left((1+a)+a(1+a)-a^2-2a(1+a)(1-a)-2a^3\right) \\
= & \frac{1}{(1-a)^3(1+a)}\left(1+a+a+a^2-a^2-2a+2a^3-2a^3\right) = \frac{1}{(1-a)^3(1+a)}.
\end{aligned}
$$

The following theorem offers us the result of orthogonality[9, 14].

**Theorem 16.** *The system* $\{\sqrt{\frac{1-a}{\pi(1+a)}}Sinc_a(\cdot - n\pi) : n \in \mathbb{Z}\}$ *is an orthonormal system.*

*Proof.* We know that

$$Sinc_a(t) = p_a(\pi t)Sinc(\pi t) = (1-a^2)\sum_{l=1}^{\infty} a^{l-1}\frac{\sin l\pi t}{\pi t}.$$

The Fourier transform of $Sinc_a$ is

$$(Sinc_a)^{\wedge}(\xi) = (1-a^2)\sum_{l=1}^{\infty} a^{l-1}\frac{1}{\sqrt{2\pi}}\chi_{[-l\pi,l\pi]}(\xi).$$

We therefore get that, for any integer $n$,

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\sqrt{\frac{1-a}{1+a}}Sinc_a(t)\sqrt{\frac{1-a}{1+a}}Sinc_a(t-n)\,dt \\
= & \frac{1-a}{1+a}\frac{(1-a^2)^2}{2\pi}\int_{-\infty}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1}\chi_{[-l\pi,l\pi]}(\xi)\chi_{[-m\pi,m\pi]}(\xi)e^{in\xi}d\xi \\
= & \frac{(1-a)^3(1+a)}{2\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1}\int_{[-l\pi,l\pi]\cap[-m\pi,m\pi]} e^{in\xi}d\xi \\
= & (1-a)^3(1+a)\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} a^{l-1}a^{m-1}min\{l,m\}\delta_{n,0} \\
= & \delta_{n,0}.
\end{aligned}
$$

# 5|A Framework for Analytical Decision-Making Applications

The Bilinear Hilbert Transform (BHT) offers a powerful framework for enhancing decision-making processes in environments where nonlinear signals[1, 10] and real-time data are critical. By decomposing complex signals, the BHT provides deeper insights, enabling more informed decisions across a variety of domains.

## 5.1|Signal Decomposition for Better Insights

The BHT's ability to decompose nonlinear signals allows decision-makers to uncover hidden patterns in data, leading to better predictions and responses. For example, in predictive maintenance, the BHT helps detect early signs of equipment failure by analyzing complex signal behaviors. Similarly, in finance, it reveals trends and volatility in stock prices, improving investment strategies.

## 5.2|Integration of Bedrosian Identity

The Bedrosian identity plays a key role in combining different signal components while maintaining their integrity. This is especially useful in models involving multiple interacting variables, such as economic forecasting, where inflation, interest rates, and currency values need to be harmonized for accurate predictions.

## 5.3|Generalized Sinc Functions for Precision

Generalized Sinc functions enhance the precision of signal decomposition in BHT, improving the accuracy of models in fields such as biomedical signal processing or industrial control. These functions allow for more reliable interpretations of sensor data, leading to optimized processes and improved decision outcomes.

## 5.4|Real-Time Decision-Making and Multi-Domain Use

The BHT's real-time processing capability makes it ideal for applications requiring immediate action, such as telecommunications or supply chain management. Its adaptability extends to multiple fields, including healthcare, finance, and environmental monitoring, where it aids in optimizing resource use and improving response times.

## 5.5|Integration with Future Technologies

By integrating BHT with machine learning, decision-making processes can become more automated and adaptive. For example, in autonomous systems, BHT enhances real-time sensor data processing, improving machine learning models' performance. As industries move toward Industry 5.0, this framework provides a way to combine human insight with machine efficiency, particularly in sustainable practices.

# 6|Conclusion

With the Hilbert transform, we define the bilinear Hilbert transform. We have demonstrated some properties of BHT.The generalized Sinc function is a special solution of $H\rho,c\ \theta_a\ t - \cdot \quad t\ /\ \rho\ t\ H,c\ s\vartheta_a\ t - \cdot \quad t$.After that, we consider some the nonlinear Sinc function, which is also the solution of this equation.Last,The nonlinear Sinc function system $\{\sqrt{\frac{1-a}{\pi(1+a)}}Sinca(\cdot - n\pi \quad n \in \mathbb{Z}\}$ is an orthonormal system.

Additionally, we have introduced a framework that extends BHT's utility into analytical decision-making applications.This framework leverages the decomposition properties of the BHT to support real-time decision-making, precision modeling, and integration with AI technologies across multiple fields. The BHT enables more accurate and responsive decision-making, particularly in environments that require the handling of complex nonlinear data and real-time signal processing. Moreover, by applying this framework to various industries such as finance, healthcare, and industrial optimization, we highlight the versatility of the BHT in improving efficiency and predictive accuracy. As industries continue to adopt AI-driven and real-time decision-making tools, the BHT framework offers an adaptable and scalable solution for future technological advancements, aligning with the principles of Industry 5.0 and sustainable practices.

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# Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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